CSCI 7000 Fall 2023: Problems on Generating Functions

Joshua A. Grochow

September 12, 2023

To work on during class:

From last week, recall we had

$$\mathcal{F}_n := \{x \in \{0,1\}^n \mid \text{all 0's come in consecutive pairs 00}\},$$

with $|\mathcal{F}_n| = F_n$, the n-th Fibonacci number (starting with $F_0 = F_1 = 1$). This has generating function $F(x) = 1/(1-x-x^2)$. (Beware - some authors start with $F_0 = 0, F_1 = 1$!)

Then we defined a related sequence of sets:

$$\mathcal{G}_n := \{x \in \{0,1,2\}^n \mid \text{all 0s come in consecutive pairs 00, and there is exactly one 2}\}.$$

1. Let $\mathcal{F} = \bigcup_{n \geq 0} \mathcal{F}_n$ be the set of all strings (of any length) where all 0s come in consecutive pairs.
   (a) Give a bijection between $\mathcal{F}$ and $\{0,1\}^*$ (the set of all binary strings).
   (b) Use this bijection to motivate a quick re-derivation of the generating function for the Fibonacci series, without using the Fibonacci recurrence. *Hint:* Start with a two-variable generating function

   $$\sum_{n \geq 0, k \geq 0} x^{n-k} y^k \cdot (\text{ \# strings with exactly } k \text{ zeros and } n-k \text{ ones}).$$

2. Use the product formula for generating functions, and the known generating function for the Fibonacci series, find the generating function for the series $G_n = |\mathcal{G}_n|$.

3. In class we derived the recurrence

$$G_n = G_{n-1} + G_{n-2} + F_{n-1}.$$

Use this to find the generating function for the sequence G_n.

4. Let $f(n, k)$ denote the number of lists $[x_1, \ldots, x_k]$, consisting of k (not necessarily distinct) non-negative integers, such that $\sum_{i=1}^{k} x_i = n$. Find a simple closed-form formula for $f(n, k)$ (hint: it will be in terms of binomial coefficients) using generating functions.

5. (a) Use operations on generating functions to prove that the Fibonacci numbers satisfy $F_0 + F_1 + \cdots + F_n = F_{n+2} - 1$ for all $n \geq 0$.

   (b) Now give a bijective proof of that fact.

To work on outside of class

1. (May be harder!) Let $H_{n,k}$ be the number of strings in $\mathcal{F}_n$ with exactly k ones. In class we derived the recurrence

$$H_{n,k} = H_{n-1,k-1} + H_{n-2,k}.$$

Using this, find the two-variable generating function for the array of numbers $H_{n,k}$. Then use operations on generating functions we learned about in class to derive the generating function for $|\mathcal{G}_n|$.